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LETTER TO THE EDITOR

State extended uncertainty relations

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Abstract. A scheme for construction of uncertainty relations for n observables and m states is presented. Several lowest-order inequalities are displayed and briefly discussed. For two states $|\psi\rangle$ and $|\phi\rangle$ and canonical observables the (entangled) extension of the Heisenberg relation reads $[\Delta p(\psi)]^2[\Delta q(\phi)]^2 + [\Delta p(\phi)]^2[\Delta q(\psi)]^2 \geqslant \frac{1}{2}$.

1. Introduction

The uncertainty principle is one of the basic principles in quantum physics. It was introduced by Heisenberg [1] on the example of the canonical observables p and q, and rigorously proved by Kennard and Weyl [1] in the form of the inequality $(\Delta p)^2(\Delta q)^2 \geqslant \frac{1}{4}$, where $(\Delta X)^2$ is the variance (dispersion) of X (for the sake of brevity we work with dimensionless observables). This inequality is known as the Heisenberg uncertainty relation (UR) for p and q. It was made more precise and extended to two arbitrary quantum observables by Schrödinger and Robertson [2] and to several observables by Robertson [3].

The Heisenberg UR became an irrevocable part of almost every textbook in quantum physics while the interest in the more precise Schrödinger [2] and Robertson [3] UR has grown up only recently in connection with the experimental generation of squeezed states of the electromagnetic field [4] and their generalization to two and several arbitrary observables [5–8]. The Robertson UR has been recently extended to all characteristic coefficients of the uncertainty matrix [9]. Extensions of the Heisenberg UR to higher moments of p and q are made in [10].

The URs listed above, and perhaps all the others so far considered, relate certain combinations of statistical moments of the observables in one quantum state. The main aim of this paper is to extend the uncertainty principle to several states. The physical idea of such an extension is simple: one can measure and compare the statistical moments of two (or more) observables not only in one and the same state, but in two (or more) different states. The Hilbert space model of quantum mechanics permits us to derive easily such *state-extended* URs. The extended URs can be divided into two classes—*entangled URs* and nonentangled URs. A UR is called state entangled if it cannot be factorized over distinct states.

Next we recall the ordinary characteristic Urs, which include the known Schrödinger and Robertson ones and then extend these URs to several states. Some other state-extended URs are also established. Finally the simplest types of extended UR are displayed and discussed briefly.

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2. Characteristic URs

The Schrödinger (or Schrödinger–Robertson) [2] UR for two observables X and Y reads

$$(\Delta X)^2(\Delta Y)^2 - (\Delta XY)^2 \geqslant \tfrac{1}{4}|\langle[X, Y]\rangle|^2 \tag{1}$$

where $\langle X\rangle$ is the mean value of X in a given state, $(\Delta X)^2 = \langle X^2\rangle - \langle X\rangle^2$ is the variance (the dispersion) of X and $\Delta XY \equiv \langle XY + YX\rangle/2 - \langle X\rangle\langle Y\rangle$ is the covariance of X and Y. This UR was derived by Schrödinger from the Schwartz inequality for the matrix element $\langle\psi|(X - \langle X\rangle)(Y - \langle Y\rangle)|\psi\rangle$. The less precise inequality $(\Delta X)^2(\Delta Y)^2 \geqslant \frac{1}{4}|\langle[X, Y]\rangle|^2$ is usually called the Heisenberg UR for X and Y.

Robertson [3] has formulated the uncertainty principle for several observables $X_1, \ldots, X_n$ in terms of an inequality between determinants of the uncertainty matrix $\sigma(\vec{X}; \psi)$ and the matrix $C(\vec{X}; \psi)$ of the mean values of commutators of X_i and X_j,

$$\det\sigma(\vec{X}; \psi) - \det C(\vec{X}; \psi) \geqslant 0 \tag{2}$$

where $\vec{X} = X_1, \ldots, X_n, \sigma_{ij} = (\frac{1}{2})\langle X_iX_j + X_jX_i\rangle - \langle X_i\rangle\langle X_j\rangle$ and $C_{jk} = -(\frac{i}{2})\langle[X_j, X_k]\rangle$. For $n = 2$ the inequality (2) recovers (1). Robertson first proved the non-negative definiteness of the matrix $R(\vec{X}; \psi)$ (to be called the Robertson matrix), $R(\vec{X}; \psi) = \sigma(\vec{X}; \psi) + \mathrm{i}C(\vec{X}; \psi) \geqslant 0$. This means that all principal minors of R are non-negative. For $n = 2$ the inequality (2) coincides with $R(\vec{X}; \psi) \geqslant 0$. Robertson URs hold for mixed states ρ as well. Recently [9] the UR (2) has been extended to all characteristic coefficients [12] of the uncertainty matrix.

In comparison with the Heisenberg UR the Schrödinger and Robertson ones have the advantage of being covariant under linear nondegenerate transformations of the observables, in particular under linear canonical transformations [8,9]. Symmetric to this is the invariance of our URs for one observable and m states (established below) under linear transformation of states.

3. State extended URs

It should be useful first to recall that the derivation of the Robertson UR resorts to the following lemma.

Lemma 1 (Robertson). *If H is a non-negative definite Hermitian matrix, then*

$$\det S - \det A \geqslant 0 \tag{3}$$

where S and A are the real and the imaginary part of H, $H = S + \mathrm{i}A$.

It is worth recalling that a matrix H is non-negative iff all its principal minors $M_r(H)$ are non-negative,

$$H \geqslant 0 \longleftrightarrow M_r(H) \geqslant 0 \qquad r = 1, 2, \ldots, n. \tag{4}$$

The proof of this lemma can be found in [3]. With minor changes in the notations it is reproduced in [11]. The Robertson UR (2) corresponds to $H = R(\vec{X}; \rho)$ in (3). In [9] this lemma was extended to all principal minors and to all characteristic coefficients $C_r^{(n)}$ of S and A,

$$C_r^{(n)}(S) - C_r^{(n)}(A) \geqslant 0 \qquad r = 1, 2, \ldots, n. \tag{5}$$

The characteristic URs of [9] correspond to $S = \sigma(\vec{X}; \rho)$ and $A = C(\vec{X}; \rho)$ in (5).

The state extensions of the ordinary URs, which we shall derive below, are based on the different physical choices of the matrix H in (3)–(5) and on the following lemma.

Lemma 2. *If H_μ are non-negative definite Hermitian $n \times n$ matrices, $\mu = 1, \ldots, m$, then*

$$C_r^{(n)}(S_1 + \cdots + S_m) - C_r^{(n)}(A_1 + \cdots + A_m) \geqslant 0 \tag{6}$$

$$C_r^{(n)}(H_1 + \cdots + H_m) - C_r^{(n)}(H_1) - \cdots - C_r^{(n)}(H_m) \geqslant 0 \tag{7}$$

where S_μ and A_μ are the real (and symmetric) and the imaginary (and antisymmetric) parts of H_μ.

Proof. The validity of (6) immediately follows from the Robertson lemma and its extension (5), and the known fact that a sum of Hermitian non-negative matrices is a Hermitian non-negative matrix. We proceed with the proof of the inequality (7). It is sufficient to establish it for $m = 2$. Let G and H be Hermitian non-negative definite matrices. We have to prove that $C_r^{(n)}(G+H) - C_r^{(n)}(G) - C_r^{(n)}(H) \geqslant 0$. Since the characteristic coefficients are the sums of all principal minors [12] it is sufficient to consider the case of $r = n$, i.e. to prove the inequality $\det(G + H) - \det G - \det H \geqslant 0$.

(a) Let one of the two matrices (say G) be positive definite. Then both G and H can be diagonalized by means of a unitary matrix U [12], $G' = U^\dagger G U = \mathrm{diag}\{g_1, \ldots, g_n\}$, $H' = U^\dagger H U = \mathrm{diag}\{h_1, \ldots, h_n\}$ and

$$\det(G + H) = \prod_i (g_i + h_i) = \prod_i g_i + \prod_i h_i + \Delta \tag{8}$$

where $\Delta = \det(G + H) - \det G - \det H = \det(G' + H') - \det G' - \det H'$, $i = 1, \ldots, n$,

$$\Delta = g_1 \prod_{j=2}^{n} h_j + g_1 g_2 \prod_{j=3}^{n} h_j + \cdots + h_1 h_2 \prod_{i=3}^{n} g_i + h_1 \prod_{i=2}^{n} g_i. \tag{9}$$

In view of $g_i > 0$ and $h_j \geqslant 0$ all terms in (9) are non-negative, thereby $\Delta \geqslant 0$.

(b) If both G and H are only non-negative definite, then at least one g_i and one h_j are vanishing, that is $\det G = 0 = \det H$ and from nonnegativity of the sum $G + H \geqslant 0$ and (8) we obtain $\det(G + H) = \Delta \geqslant 0$. End of the proof.

□

Remark 1. From the above proof it follows that if $\det \sum H_\mu = \sum \det H_\mu$ then $\det H_\mu = 0$, the inverse being untrue.

Equations (6) and (7) can be called *extended characteristic inequalities*. They are invariant under the similarity transformation of the matrices H_μ. At $m = 1$ (one state) they recover the relations (5).

By a suitable physical choice of the non-negative Hermitian matrices H_μ in the inequalities (4), (6), (7) one can obtain a variety of URs for several states and observables. We point out three physical choices of matrices H_μ,

$$H = R(\vec{X}; \rho) = \sigma(\vec{X}; \rho) + \mathrm{i}C(\vec{X}; \rho) \qquad \text{(Robertson matrix)} \tag{10}$$

$$H = \Gamma(\chi_1, \ldots, \chi_n) = R(\vec{X}; \vec{\psi}) \qquad \|\chi_k\rangle = (X_k - \langle\psi_k|X_k|\psi_k\rangle)|\psi_k\rangle \tag{11}$$

$$H = \Gamma(\phi_1, \ldots, \phi_n) = G(\vec{X}; \vec{\psi}) \qquad \|\phi_k\rangle = X_k|\psi_k\rangle \tag{12}$$

where Γ is the Gram matrix, $\Gamma_{ij}(\Phi_1, \ldots, \Phi_n) = \langle\Phi_i\|\Phi_j\rangle$, and $|\psi_k\rangle$ are normalized pure states. The diagonal elements $R_{ii}(\vec{X}; \vec{\psi})$ and $R_{ii}(\vec{X}; \rho)$ are just the variances of X_i in the state $|\psi_i\rangle$ and (generally mixed) state ρ, while $G_{ii} = \Gamma_{ii}(\phi_1, \ldots, \phi_n) = (\Delta X_i(\psi_i))^2 + \langle\psi_i|X|\psi_i\rangle^2$. Therefore the inequalities obtained in the above scheme can be regarded as *state-extended URs*. For brevity URs for n observables and m states should be called URs of type *(n,m)*.

For pure states $|\psi_k\rangle$ (10) is a particular case of (11), and the common structure of (10)–(12) is $H = \Gamma(\Phi_1, \dots, \Phi_n) = T(\vec{X}, \vec{\psi})$, where Φ_k denote the corresponding nonnormalized states $\|\Phi_k\rangle$. Let us note that $\Gamma(\Phi_1, \Phi_2) \geqslant 0$ is equivalent to the Schwartz inequality. For one observable X the matrix $G(X, \vec{\psi})$ is covariant under linear transformation of states,

$$|\psi_i'\rangle = U_{ik}^*|\psi_k\rangle \rightarrow G(\vec{X}, \vec{\psi}') = UG(\vec{X}, \vec{\psi})U^\dagger.$$

This property entails the invariance of the equality in the extended highest-order characteristic UR of type $(1, m)$, constructed by means of $G(X, \vec{\psi})$. If $UU^\dagger = 1$ then all order extended characteristic URs of type $(1, m)$ are invariant. Compare this symmetry with that of ordinary characteristic URs under linear transformations of observables [8, 9]. The extended URs of types (n, m) with $n > 1$ do not possess such symmetry.

In all three cases of H with pure states the URs $H \geqslant 0$ are disentangled by means of linear transformations of states. The URs corresponding to (6) and (7) are state entangled. The proof of the nonentangled character of the URs $H \geqslant 0$ for (10)–(12) with pure states can be easily carried out using the diagonalization of $\Gamma = \Gamma(\Phi_1, \dots, \Phi_m)$.

4. Extended URs of simplest types

URs of type (1,2). For $m = 2$ (two states) the choices $H = R(X; \vec{\psi})$ and $H = G(X; \vec{\psi})$ in (4), (6) and (7) produce two different URs, which we write down as

$$\Delta X(\psi_1))^2(\Delta X(\psi_2))^2 \geqslant |\langle\psi_1|(X - \langle\psi_1|X|\psi_1\rangle)(X - \langle\psi_2|X|\psi_2\rangle)|\psi_2\rangle|^2 \tag{13}$$

$$((\Delta X(\psi_1))^2 + \langle\psi_1|X|\psi_1\rangle^2)((\Delta X(\psi_2))^2 + \langle\psi_2|X|\psi_2\rangle^2) \geqslant |\langle\psi_1|X^2|\psi_2\rangle|^2. \tag{14}$$

Since the right-hand sides of (13) and (14) are generally greater than zero these inequalities reveal correlations between the statistical second moments of X in different states.

These two URs are independent in the sense that none of them is more precise than the other. To prove this suffice it to consider the example of $X = p$ and two Glauber coherent states. The minimization of (13) and (14) occurs iff the two nonnormalized states in the Gram matrix are proportional. In the case of (13) this is $(X - \langle 2|X|2\rangle)|\psi_2\rangle = \lambda(X - \langle 1|X|1\rangle)|\psi_1\rangle$ wherefrom we easily deduce that if X is a continuous observable (such as q, p or $p^2 - q^2$ and $pq + qp$) then UR (13) is minimized iff $|\psi_1\rangle = |\psi_2\rangle$. It follows from this condition that (13) and (14) can be used for construction of distances between quantum states (observable induced distances) [11].

URs of type (2,1). For $n = 2$ the inequalities (4), (6) and (7) coincide. For two observables X, Y and one state the Robertson choice (10) coincides with (11) and when replaced in (4)–(7) it produces the Schrödinger UR (1). The choice (12) in (6) and (7) generates the invariant UR

$$[(\Delta X)^2 + \langle X\rangle^2][(\Delta Y)^2 + \langle Y\rangle^2)] \geqslant (\Delta XY + \langle X\rangle\langle Y\rangle)^2 + \tfrac{1}{4}|\langle[X, Y]\rangle|^2 \tag{15}$$

which however can be shown to be less precise than the Schrödinger one (1). The interpretation of any UR of type (2, 1) is the same as that of the Schrödinger UR.

URs of type (2,2). The number of possible URs of type (2, 2) is much greater. The inequalities $R(X, Y; \psi_1, \psi_2) \geqslant 0$ and $G(X, Y; \psi_1, \psi_2) \geqslant 0$ can be displayed as ($\langle i|X|i\rangle \equiv \langle\psi_i|X|\psi_i\rangle$)

$$(\Delta X(\psi_1))^2(\Delta Y(\psi_2))^2 \geqslant |\langle\psi_1|(X - \langle 1|X|1\rangle)(Y - \langle 2|Y|2\rangle)|\psi_2\rangle|^2 \tag{16}$$

$$[(\Delta X(\psi_1))^2 + \langle 1|X|1\rangle^2][(\Delta Y(\psi_2))^2 + \langle 2|Y|2\rangle^2] \geqslant |\langle\psi_1|XY|\psi_2\rangle|^2. \tag{17}$$

It is not difficult to establish (after some manipulations) that the inequality (17) is less precise than (16). The equalities in (16) and (17) are not invariant under linear transformations of observables and/or states. The equations (6) and (7) with $H_1 = R(X, Y; \psi_1)$ and $H_2 = R(X, Y; \psi_2)$ both produce an entangled but very compact (2, 2) UR,

$$\tfrac{1}{2}[(\Delta X(\psi_1))^2(\Delta Y(\psi_2))^2 + (\Delta X(\psi_2))^2(\Delta Y(\psi_1))^2] - \Delta XY(\psi_1)\Delta XY(\psi_2) \geqslant \tfrac{1}{4}\langle\psi_1|[X, Y]|\psi_1\rangle\langle\psi_2|[X, Y]|\psi_2\rangle^*. \quad (18)$$

The equality in this relation is invariant under linear transformations of X, Y, but not of $|\psi_1\rangle$ and $|\psi_2\rangle$. With $|\psi_1\rangle = |\psi_2\rangle$ in (18) one recovers the Schrödinger UR (1). The inequality (18) should be referred to as the *state-extended Schrödinger UR*. For the canonical p and q it simplifies to

$$\tfrac{1}{2}[(\Delta p(\psi_1))^2(\Delta q(\psi_2))^2 + (\Delta p(\psi_2))^2(\Delta q(\psi_1))^2] - \Delta pq(\psi_1)\Delta pq(\psi_2) \geqslant \tfrac{1}{4}. \quad (19)$$

Similar to (but less precise than) (18) is the (2, 2) UR obtained again from (6) and (7) with the third choice (12). The entangled UR (18) admits a less precise version of the form (corresponding to $\Delta XY = 0$)

$$\tfrac{1}{2}[(\Delta X(\psi_1))^2(\Delta Y(\psi_2))^2 + (\Delta X(\psi_2))^2(\Delta Y(\psi_1))^2] \geqslant \tfrac{1}{4}|\langle\psi_1|[X, Y]|\psi_1\rangle\langle\psi_2|[X, Y]|\psi_2\rangle|. \quad (20)$$

The latter inequality can be regarded as an *entangled extension of the Heisenberg UR*. For $X = p$ and $Y = q$ the right-hand side of (20) simplifies to $\frac{1}{4}$.

In view of the remark 1 if UR (18) is minimized then (1) is saturated by $|\psi_1\rangle$ and $|\psi_2\rangle$. Therefore (18) can be used for finer classification of Schrödinger intelligent states. From any extended UR one can obtain new ordinary UR by fixing all but one of the states. For example if $|\psi_1\rangle$ in (19) is fixed as a canonical coherent state then (19) produces $(\Delta p)^2 + (\Delta q)^2 \geqslant 1$. The latter UR is minimized in canonical coherent states only, while the Heisenberg UR $(\Delta p)^2(\Delta q)^2 \geqslant \frac{1}{4}$ is minimized in any squeezed state with $\Delta pq = 0$.

5. Conclusion

We have established several classes of extended characteristic URs of type (n, m) for n observables and m states using the Gram matrices of suitably constructed nonnormalized states. Entangled URs can be obtained using characteristic inequalities (6) and (7).

The extended URs reveal global statistical correlations of quantum observables in distinct states. The characteristic inequalities could be useful in many fields of mathematical and quantum physics, in particular in precise measurement theory. Extended URs can be used for construction of observable induced distances between quantum states and for finer classification of states, in particular of group-related coherent states.

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